Pergamon
www.elsevier.com/locate/jappmathmech
PII: S0021–8928(00)00045-9
J. Appl. Maths Mechs, Vol. 64, No. 2, pp. 229–242, 2000

0021–8928/00/$—see front matter

THE FOURTH ALGEBRAIC INTEGRAL OF KIRCHHOFF'S EQUATIONS†

S. T. SADETOV

Rostov-on-Don

e-mail: ssadetov@uic2.rnd.runnet.ru

(*Received* 4 *November* 1999)

The problem proposed by Steklov [1] of finding all the cases when the equations of motion of a rigid body in an ideal fluid allow of a fourth integral in the form of a homogeneous polynomial of arbitrary degree is considered. When there is a certain symmetry, when other methods do not work [2–6], this problem is solved, including for a particular integral: all these cases are exhausted by the classical cases. An improvement of Husson's approach [7] is proposed, beginning from the second step.

1. INTRODUCTION AND FORMULATION OF THE RESULT

The equations of inertial motion of a rigid body for irrotational flow around the body of an ideal homogeneous incompressible fluid, which is at a rest at infinity, have the form [8]

$$\begin{aligned} &\dot{M} = M\times\omega + p\times v, \quad \dot{p} = p\times\omega, \quad \omega = \partial T/\partial M, \quad v = \partial T/\partial p \\ &2T = \langle aM, M\rangle + 2\langle bM, p\rangle + \langle cp, p\rangle \end{aligned} \tag{1.1}$$

Here M has the meaning of the total moment of the "body plus fluid" system, p is the overall momentum, differentiation with respect to time is carried out in a system of coordinates frozen in the body, a, b and c are constant 3×3 matrices, comprising the positive-definite matrix

$$\begin{Vmatrix} a & b \\ b & c \end{Vmatrix}$$

inverse to the matrix of the inertia coefficients when the added masses are taken into account, T is the overall kinetic energy, ω is the angular velocity of the body and v is the velocity of the origin of coordinates [9].

Equations (1.1) have an invariant measure and three quadratic conservation laws, discovered by Kirchhoff: T, $\langle M, p\rangle$, p^2.

These equations also arise in other physical problems. For example, they describe the rotation around a fixed point of an electrically charged rigid body in a uniform magnetic field and an axisymmetic force filed with a quadratic potential, neglecting the effect of self-induction (in this case only the form $\langle aM, M\rangle$) is positive-definite)[10, 11].

The function $F(x)$ is said to be algebraic at the point $x = 0$, if $A_0 + A_1F + \ldots + A_kF^k = 0$, where $A_0(x), \ldots, A_k(x)$ are functions that are analytic at the point $x = 0$, $A_k(x) \neq 0$, $k \in N$, see [12].

The integral of system (1.1) is said to be supplementary [1, 8], if it is independent of the classical integral and is said to be particular [13, 14] if it is only conserved when $\langle M, p\rangle = 0$. It can be shown that the additional integral (general or particular) of system (1.1), algebraic at the point $M = p = 0$, can be reduced to a corresponding supplementary homogeneous rational integral [15, 2].

Steklov [1] and Lyapunov [16] obtained all the cases of the existence of a supplementary linear and quadratic integral of system (1.1)(later this was done in [17] for an arbitrary, not necessarily positive-definite, form T).

Theorem. For values of the parameters

$$a = \operatorname{diag}(a_1, a_1, a_3), \ a_1 > 0, \ a_3 > 0, \ b = \operatorname{diag}(b_1, b_1, b_3), \ c = \operatorname{diag}(c_1, c_2, c_3)$$

all cases of the existence of an additional algebraic general or particular integral of system (1.1) are exhausted by the classical integrals: Kirchhoff [8], Clebsch [18] and Chaplygin [13].

†*Prikl. Mat. Mekh.* Vol. 64, No. 2, pp. 237–251, 2000.

Corollary. Chaplygin's integral of the fourth degree [13] is not continued to the algebraic integral outside the surface $\langle M, p\rangle = 0$.

This problem was investigated in [2] when $a_1 \neq a_2 \neq a_3$ and for arbitrary b and c by making the replacement $p \to \varepsilon p$ and splitting the separatrice of Euler's case, where the Hamiltonian nature of the perturbed problem (1.1) is essentially employed, see [3, 4].

When $a_1 = a_2$ in the unperturbed case of a regular Euler–Lagrange precession there are no separatrices. The splitting of the other separatrices with another introduction of a small parameter is established either for small $c_1 + c_2 \neq 0$ [5], or for fairly large a_3/a_1 [6]; the density of the secular set—for almost all $a_3/a_t > 2$ [6] (with the exception of the classical integrable cases).

Remark. In the most important case when the body has three mutually perpendicular planes of symmetry

$$a = \operatorname{diag}(a_1, a_2, a_3), a_i > 0,\ \ i = 1,2,3,\ \ b = 0,\ \ c = \operatorname{diag}(c_1, c_2, c_3)$$

the theorem of this paper, together with the result obtained in [2], concludes the problem formulated by Steklov [1].

The proof of the theorem rests on Husson's method [7] of the expansion of the supplementary integral in a small parameter, uses the technique of the analysis of algebraic relations of Abel integrals, rising to the classical Abel and Chebyshev results [19] and which rest on the estimate of the algebraic multiplicity of zero defined in Section 1.

When the unperturbed system has a complete set of algebraic integrals (Section 5), the obstacles to integrability in osculating algebraic variables are reduced directly to the residues and periods on the unperturbed solution in certain integro-differential operators of the vector field (cf. [20, 14]). These operators contain an integration along the unperturbed solution and differentiation with respect to the osculating variables and the perturbation parameter.

2. THE ALGEBRAIC MULTIPLICITY OF ZERO

Suppose $\varkappa \in \mathbf{Q}$. We will but

$$W = s^2 - 1,\ \ \vartheta = (s-1)/(s+1),\ \ \mathbf{M} = N_0 \cup (-3/2 - N_0),\ \ N = (0, 1, 2\ldots)$$

Abel's lemma [21]. The integral $\int W^{\varkappa} ds$ is an algebraic function of s, $\ln \vartheta$ if and only if $\varkappa \in \mathbf{Z}/2$. In particular, it will be an algebraic function of s if and only if $\varkappa \in \mathbf{M}$.

The proof of Abel's lemma follows from the well-known results in [21, 22].

Suppose a ring K is generated above C by the functions s, $1/\sqrt{W}$ and a ring L is generated above K by the function in ϑ. For

$$f = \sum_{l=0}^{n} f_l (\ln \vartheta)^l,\ \ f_l \in K + W^{1/4} K,\ \ n \in N_0$$

we put $\nu_\infty f := \min_l \operatorname{ord}_\infty f_l$.

The following two properties of $\operatorname{ord}_\infty$ are extended to ν_∞.

Property 1.

$$\nu_\infty(f + g) \geqslant \min\{\nu_\infty f, \nu_\infty g\}$$

When $\nu_\infty f \neq \nu_\infty g$ we have an equality.

Property 2.

$$\nu_\infty(fg) = \nu_\infty f + \nu_\infty g$$

Lemma 1. Suppose $g(s)$ is a function that is algebraic at the point $s = \infty$. Then
(a) if $\operatorname{ord}_\infty g \neq 0$, then $\operatorname{ord}_\infty g' = \operatorname{ord}_\infty g + l$;
(b) if $\operatorname{ord}_\infty g = 0$, then $\operatorname{ord}_\infty g' > 1$.

Property 3. Suppose $f \in L$. Then
(a) $\nu_\infty f' \geqslant 1 + \nu_\infty f$;
(b) if $\nu_\infty f' \leqslant 1$, then $\nu_\infty f' = 1 + \nu_\infty f$.

Criterion 1. Suppose $f \in L$, $v_\infty f' \geqslant 2$ Then

$$\int f ds \in L, \quad v_\infty\left(\int f ds\right) \geqslant 0$$

Criterion 2. Suppose

$$J_{lm} = \int \left(W^l \int W^m ds\right) ds, \qquad l, m \in Z$$

Then the condition $J_{lm} \notin L$ is equivalent to $\min\{l, m\} \leqslant -1 \leqslant l + m$.

The proofs of properties 1, 2, and 3 of Lemma 1 and Criteria 1 and 2 are given in Section 6.

3. HUSSON VARIABLES

We make the replacement

$$\begin{aligned}
&(M, p) \to (y_1, y_2, r, z_1, z_2, p_3), \qquad t \to it \\
&\omega = (p, q, r), \qquad p = (p_1, p_2, p_3) \\
&y_1 = p + iq, \quad y_2 = p - iq, \quad z_1 = p_1 + ip_2, \quad z_2 = p_1 - ip_2
\end{aligned}$$

used by Husson for a symmetrical heavy rigid body (see [23]). We obtain

$$\begin{aligned}
\dot{y}_1 &= -\alpha r y_1 + p_3(\beta_3 z_1 + \beta_2 z_2)/2 - \beta(1-\alpha)(r z_1 + y_1 p_3) + 2\beta_1 r z_2 \\
\dot{y}_2 &= \alpha r y_2 - p_3(\beta_2 z_1 + \beta_3 z_2)/2 + \beta(1-\alpha)(r z_2 + y_2 p_3) - 2\beta_1 r z_1 \\
\dot{r} &= \frac{\beta_2}{4(1-\alpha)}(z_1^2 - z_2^2) + \frac{\beta}{2}(y_2 z_1 - y_1 z_2) - \frac{\beta_1}{1-\alpha}(y_2 z_2 - y_1 z_1) \\
\dot{z}_1 &= y_1 p_3 - r z_1, \quad \dot{z}_2 = r z_2 - y_2 p_3, \quad \dot{p}_3 = (y_2 z_1 - y_1 z_2)/2
\end{aligned} \tag{3.1}$$

where

$$\begin{aligned}
&1 - \alpha = a_1 / a_3, \quad 2\beta = 2b_3 - b_1 - b_2, \quad 2\beta_1 = b_1 - b_2 \\
&\beta_2 = a_1(c_1 - c_2), \quad \beta_3 = a_1(c_1 + c_2 - 2\tilde{c}_3), \quad \tilde{c}_3 = c_3 - \beta^2 / a_3 + \beta_1^2 / a_1
\end{aligned}$$

We will write the linear combinations of the initial integrals

$$H = 2a_1 T + (\beta_1^2 - a_1(c_1 + c_2)/2)p^2, \quad H_1 = 2a_1\langle M, p\rangle - 2\beta p^2, \quad H_2 = p^2$$

in the form

$$\begin{aligned}
H &= y_1 y_2 + (1-\alpha) r^2 + \beta_2(z_1^2 + z_2^2)/4 - \beta_3 p_3^2 / 2 \\
H_1 &= y_1 z_2 + y_2 z_1 + 2(1-\alpha) r p_3 - \beta_1(z_1^2 + z_2^2) - 2\beta z_1 z_2 - 2(2-\alpha)\beta p_3^2 \\
H_2 &= z_1 z_2 + p_3^2
\end{aligned} \tag{3.2}$$

When $\beta_2 \neq 0$ (i.e. with the exception of the integrable Kirchhoff case), without loss of generality we can put $\beta_2 = 1$, assuming that either $p, b \in \mathbf{R}$ or $p, b \in i\mathbf{R}^3$ (the latter is equivalent to replacing p, b and c by $p/\sqrt{\beta_2}$, $b\sqrt{\beta_2}$ and $c\beta_2$).

4. THE REPLACEMENT $(y_2, z_1, p_3) \to \varepsilon(y_1, z_1, p_3)$

Following Husson, when $\beta = 0$ we introduce a small parameter ε into the system by means of this replacement.

The system and the integrals take the form

$$\begin{aligned}
\dot{y}_1 &= -\alpha r y_1 + p_3(\varepsilon\beta_3 z_1 + z_2)/2 - \beta(1-\alpha)(r z_1 + \varepsilon y_1 p_3) \\
\dot{y}_2 &= \alpha r y_2 - p_3(\varepsilon z_1 + \beta_3 z_2)/2 + \beta(1-\alpha)(r z_2 + \varepsilon y_2 p_3)
\end{aligned} \tag{4.1}$$

$$\begin{aligned}
&\dot{r}=\frac{1}{4(1-\alpha)}(\varepsilon^2 z_1^2-z_2^2)+\frac{1}{2}\varepsilon\beta(y_2z_1-y_1z_2)\\
&\dot{z}_1=\varepsilon y_1p_3-rz_1,\quad \dot{z}_2=rz_2-\varepsilon y_2p_3,\quad \dot{p}_3=(y_2z_1-y_1z_2)/2\\
&z_2^2/4+(1-\alpha)r^2+\varepsilon y_1y_2+\varepsilon^2(z_1^2/4-\beta_3p_3^2/2)=h\\
&y_1z_2+y_2z_1+2(1-\alpha)rp_3-2\beta h_2-2\varepsilon(1-\alpha)\beta p_3^2=h_1\\
&z_1z_2+\varepsilon p_3^2=h_2
\end{aligned}\tag{4.2}$$

Here h, h_1 and h_2 are the constants of the integrals.

We will assume that system (1.1) and, consequently, also (4.1) when $\varepsilon = 1$, has a supplementary homogeneous rational integral $\mathcal{F}(y_1, y_2, r, z_1, z_2, p_3)$ (possibly, only on the surface $\langle M, p\rangle = 0$, i.e. when $(H_1 + 2\beta H_2)(y_1, y_2, r, z_1, z_2, p_3) = 0$. Then, $\mathcal{F}(\varepsilon y_1, y_2, r, \varepsilon z_1, z_2, \varepsilon p_3)$ is a homogeneous rational integral of system (4.1) for arbitrary ε (on the surface $\varepsilon^{-1}(H_1 + 2\beta H_2)(\varepsilon y_1, y_2, r, \varepsilon z_1, z_2, \varepsilon p_3) = h_1 + 2\beta h_2 = 0$)

By virtue of relations (4.2) we can write successively

$$\begin{aligned}
&z_2=2[w-\varepsilon y_1y_2-\varepsilon^2(z_1^2/4-\beta_3p_3^2/2)]^{1/2},\quad z_1=(h_2-\varepsilon p_3^2)z_2^{-1}\\
&y_1=-(y_2z_1+2(1-\alpha)rp_3-h_1-2\beta h_2-2\varepsilon(1-\alpha)\beta p_3^2)z_2^{-1}
\end{aligned}\tag{4.3}$$

where $w = h - (1 - \alpha)r^2$. Hence, in variables

$$u=(y_2-\beta z_2)z_2^{-\alpha},\quad R=w^{(\alpha-1)/2}p_3$$

system (4.1) takes the form

$$\begin{aligned}
&\frac{du}{dr}=\frac{\varepsilon}{\rho}p_3\left[\frac{1}{2}\beta_4z_2^{1-\alpha}+\alpha u^2z_2^{\alpha-1}+2\beta u-\frac{1}{2}\varepsilon(h_2-\varepsilon w^{1-\alpha}R^2)z_2^{-1-\alpha}\right]=\\
&=\varepsilon f_1(h,h_1,h_2,u,R,r)+\varepsilon^2 f_2(h,h_1,h_2,u,R,r)+\ldots\\
&w^{(1-\alpha)/2}\frac{dR}{dr}=\frac{S}{\rho}+(1-\alpha)^2rw^{-1}p_3=g_0(h,h_1,h_2,u,r)+\varepsilon g_1(h,h_1,h_2,u,R,r)+\ldots\\
&\rho=-\frac{z_2^2}{4(1-\alpha)}+\varepsilon\beta S+\varepsilon^2\frac{(h_2-\varepsilon p_3^2)z_2^{-2}}{4(1-\alpha)},\quad \beta_4=2(2-\alpha)\beta^2-\beta_3\\
&S=h_2uz_2^{\alpha-1}+(1-\alpha)rp_3-h_1/2-\varepsilon p_3^2[(2-\alpha)\beta+uz_2^{\alpha-1}]
\end{aligned}\tag{4.4}$$

The following proposition follows from relation (4.4) and recurrent formulae (4.3).

Proposition 1. The functions f_i, g_i ($i = 0, 1, 2, \ldots$) are polynomials of h_1, h_2, u and R.

Proof. The expansion of the numerators and denominators of the right-hand sides of system (4.4) and the right-hand sides of recurrence relations (4.3) in series in powers of ε are polynomials in $h_1, h_2, u, R, y_2, p_3, y_1, z_1$. So also are the formulae $y_2(u, z_2)$ and $p_3(h, R, r)$. In view of the arbitrariness of h

$$z_2|_{\varepsilon=0}=2w=2(h-(1-\alpha)r^2)\neq 0$$

But only z_2 and w are not under the natural power. Induction with respect to i concludes the proof.

Integrating system (4.4) with $\varepsilon = 0$, we have

$$\begin{aligned}
&\tilde{u}=2^{-\alpha}C,\quad \tilde{R}=D+I+CJ\\
&I=\frac{1}{2}(1-\alpha)h_1\int w^{(\alpha-3)/2}dr,\quad J=-\frac{1}{2}(1-\alpha)h_2\int w^{\alpha-2}dr
\end{aligned}\tag{4.5}$$

where C and D are integration constants. We will assume that C and D are chosen as integration constants of system (4.4) for arbitrary ε.

In all cases below we assume $h \neq 0$.

1. *The case* $\alpha \notin \mathbf{Q}$, $(h_1, h_2) \neq 0$. It follows from the existence of four independent algebraic integrals of system (4.1) that there is an algebraic relation on $\tilde{y}_2(r)$, $\tilde{p}_3(r)$, r—the components of the solution of system (4.1) when $\varepsilon = 0$, whence the algebraic relation r, w^α, R for arbitrary C and D follows. From this relation, by virtue of the first Abel lemma ([23, 9.2]), we obtain $\alpha \in \mathbf{Q}$, i.e. a contradiction.

2. *The case* $\alpha \in \mathbf{Q}$, then $u(y_2, z_2)$, $R(p_3, r)$ are algebraic functions and the integral

$$\mathcal{F}(\varepsilon y_1, y_2, r, \varepsilon z_1, z_2, \varepsilon p_3) = \sum_{l=l_0}^{\infty} \varepsilon^l F^l(h, h_1, h_2, u, R, r)$$

is an algebraic function of h, h_1, h_2, u, R, r, ε. Suppose m is the minimum such that the coefficient F^m is functionally independent of h, h_1 and h_2. Then the integral $\Sigma_{l=l_0}^{\infty}\varepsilon^l F^l$ can be replaced by $\Sigma_{l=m}^{\infty}\varepsilon^{l-m}F^l$, which is functionally independent of h, h_1 and h_2 when $\varepsilon = 0$.

Definition. We will call the function

$$F(h, h_1, h_2, u, R, r, \varepsilon) = F^0(h, h_1, h_2, u, R, r) + \ldots + \varepsilon^n F^n(h, h_1, h_2, u, R, r)$$

the ε^n-integral of system (4.4) if, by virtue of it $dF/dr = o(\varepsilon^n)$. This integral will be said to be supplementary if F^0 is not a function solely of h, h_1 and h_2.

When $\alpha \in \mathrm{Q}$ system (4.4) has a supplementary algebraic ε^0-integral u.

Proposition 2. With the exception of the cases

$$1)\ \alpha \in \tfrac{1}{2} - N_0,\quad h_1 = 0;\ 2)\ \alpha \in -2N_0,\ h_2 = 0;\ \ 3)\ h_1 = h_2 = 0 \tag{4.6}$$

the supplementary algebraic ε^0-integral is uniquely defined, apart from a functional relationship.

Proof. In the case of two supplementary independent algebraic ε^0-integrals, all the solutions (4.5) must be algebraic curves in u, R, r space for arbitrary C and D. Consequently, I and J must be algebraic functions of r. Hence, using Abel's lemma, we have one of the following possibilities

$$1)\ \alpha = 1;\quad 2)\ h_1 = h_2 = 0;\quad 3)\ h_1 = 0,\quad \alpha - 2 \in \mathbf{M};$$
$$4)\ h_2 = 0,\quad (\alpha - 3)/2 \in \mathbf{M};\quad 5)\ \alpha - 2,\quad (\alpha - 3)/2 \in \mathbf{M}$$

Assuming $\alpha > 1$, we obtain the required result.

In cases 3–9 considered below, this condition is satisfied uniquely.

We will seek a general solution of system (4.4) in the form of series

$$u(h, h_1, h_2, C, D, r, \varepsilon) = \tilde{u} + \varepsilon u' + \varepsilon^2 u'' + \ldots,$$
$$R(h, h_1, h_2, C, D, r, \varepsilon) = \tilde{R} + \varepsilon R' + \varepsilon^2 R'' + \ldots$$

equating coefficients of powers of ε. Integrating the system obtained by equating the coefficients of ε^1, we have

$$u' = \int^r \varkappa \tilde{R}\, dr \tag{4.7}$$

$$\varkappa = \sum_{i=0}^{2} \varkappa_i C^i \equiv (\alpha - 1)2^{-\alpha}\left\{\beta_4 w^{-\alpha} + 2\beta w^{-(\alpha+1)/2} C + \frac{1}{2}\alpha w^{-1} C^2\right\}$$

$$\frac{du''}{dr} = \frac{\partial f_1}{\partial u}(h, h_1, h_2, \tilde{u}, \tilde{R}, r)u' + \frac{\partial f_1}{\partial R}(h, h_1, h_2, \tilde{u}, \tilde{u}, \tilde{R}, r)R' + f_2(h, h_1, h_2, \tilde{u}, \tilde{R}, r) \tag{4.8}$$

Proposition 3. The functions $u(h_1, h_2, C, D, r)\, R'$, u'', R'', ... are polynomials of h_1, h_2, C and D.

The proof follows from the triangularity of system (4.4), formulae (4.4) and proposition 1.

The subcase of the uniqueness of the supplementary algebraic integral when $\varepsilon = 0$. Substituting into the supplementary algebraic ε^1-integral

$$u + \varepsilon F^1(h, h_1, h_2, u, R, r)$$

the expansion of the general solution in ε, we obtain the algebraic relation

$$u' + F^1(h, h_1, h_2, \tilde{u}, \tilde{R}, r) = \mathrm{const}(h, h_1, h_2, C, D) \tag{4.9}$$

Similarly, the presence of the ε^2-integral

$$u + \varepsilon F^1(h, h_1, h_2, u, R, r) + \varepsilon^2 F^2(h, h_1, h_2, u, R, r)$$

is equivalent to the supplementary algebraic relationship

$$u'' + \frac{\partial F^1}{\partial u}(h, h_1, h_2, \tilde{u}, \tilde{R}, r)u' + \frac{\partial F^1}{\partial R}(h, h_1, h_2, \tilde{R}, r)R' + F^2(h, h_1, h_2, \tilde{u}, \tilde{R}, r) = \mathrm{const}(h, h_1, h_2, C, D) \tag{4.10}$$

Proposition 4. The following relation holds

$$F^1 = \lambda(u)R^2 + \mu(u, r)R + \nu(u, r)$$

where λ, μ, and ν, are algebraic functions of u and r (for brevity, the dependence on h, h_1 and h_2 is not shown here and below).

Proof. The functions $\tilde{R}(C, D, r)$ and $u'(C, D, r)$ are polynomials of no higher than the first degree in D. Differentiating the Husson identity (4.9) in the variables C, D and r with respect to D, we obtain

$$\int \varkappa dr + \frac{\partial F^1}{\partial R}(\tilde{u}, \tilde{R}, r) = \mathrm{const}(C, D), \quad \frac{\partial^2 F^1}{\partial R^2}(\tilde{u}, \tilde{R}, r) = \mathrm{const}(C, D) \tag{4.11}$$

It follows from the second identity of (4.11) that $\partial^2 F^1(u, R, r)/\partial R^2$ is an algebraic ε^0-integral of system (4.4), which is functionally dependent on h, h_1 and h_2, and by virtue of the proposition, on the subcase, which proves Proposition 4.

Substituting the expression of F^1 into the first identity of (4.11) we have

$$\sum_{i=0}^{2} C^i \int \varkappa_i dr + 2\lambda(2^{-\alpha}C)(I + CJ) + \mu = \mathrm{const}(C) \tag{4.12}$$

3. *The case* Q $\in \alpha \notin Z/2$, $(h_1, h_2) \neq 0$. This is the case of the uniqueness of the supplementary algebraic integral, and hence, by virtue of Proposition 4, we have formula (4.12).

We substitute the Puiseux series of the functions λ and μ at the point $C = \infty$ into (4.12)

$$2\lambda(2^{-\alpha}C) = \sum_{l=-\infty}^{l_0} \lambda_l C^l, \quad \mu(2^{-\alpha}C, r) = \sum_{m=-\infty}^{m_0} \mu_m(r) C^m$$

where $\lambda_{l0} \neq 0$ when $l \neq 0$.

When $\lambda \equiv 0$ or $l_0 \leqslant 0$ or $h_2 = 0$, $l_0 \leqslant 1$, considering the coefficient of C^2 in (4.12) we obtain that $\int \varkappa_2 dr$ is algebraic with respect to r, whence $\alpha = 0$, a contradiction.

When $\lambda_0 \geqslant 1$, $h_2 \neq 0$ considering the coefficient of C^{l_0+1}, we obtain that the expression $\delta_{l_0 1} \int \varkappa_2 dr + \lambda_{l0} J$ is algebraic with respect to r, where δ_{ij} is the Kronecker delta, whence, by virtue of Abel's lemma, $\alpha - 2 \in Z/2$, a contradiction.

When $\lambda_0 \geqslant 2$, $h_2 = 0$, $h_1 \neq 0$, consideration of the coefficient of C^{l_0}. similarly leads to a contradiction.

4. *The case* $\alpha \in 1/2 - N_0$, $h_1 \neq 0$. By Abel's lemma, all terms on the left-hand side of identity (4.12), apart from the second and fourth, are taken in the algebraic functions and logarithms; but, individually, neither the second nor the fourth are taken. Consequently, either $\beta = h_1 = 0$ (which contradicts the condition of the case) or $2\lambda = \lambda_1 C$. But the latter possibility is eliminated, since then the coefficient of C^2 in identity (4.12) would have the form

$$2^{-\alpha-1}\alpha(\alpha - 1)\int w^{-1} dr + \lambda_1 J + \mu_2(r) = \mathrm{const}$$

where $\alpha \neq 0$, while the second integral is an algebraic function of r.

Consequently, the Chaplygin case cannot be extended to the case of general integrability.

5. *The case* $\alpha \in -2N_0$, $h_2\beta \neq 0$. In this case all terms of identity (4.12), apart from $\int \varkappa_2 dr$, can be represented in the form of the sum of monomials $\ln \vartheta$, rw^l, $rw^{l+1/2}$, $l \in Z$ with constant coefficients $\vartheta = (s-1)/(s+1)$, $s = -r\sqrt{(1-\alpha)/h}$, while the expansion of this term contains $\beta \ln \sigma$, $\sigma = \sqrt{\alpha - 1}r + w^{1/2}$, a contradiction.

6. *The case* $\alpha \in -1-2N_0$, $h_1\beta_4 \neq 0$. We will use the fact that $\tilde{R} \in L$, while the coefficient of C^0D^0 in u' is equal to

$$2^{-\alpha}(\alpha-1)\beta_4 \int w^{-\alpha} I dr$$

and contains $\ln w$. The latter follows from Criterion 2 in view of the fact that -1 is contained in the integer interval $\{(\alpha-3)/2, -(\alpha+3)/2\}$. We obtain a contradiction with the first Husson equality.

7. *The case when* $\alpha \in -N_0$ or $\alpha \in -1/2 - N_0$, $\beta = h_1 = 0$. From the formulae for solutions (4.5) and (4.7) and the recurrence relations (4.3) we obtain

$$\begin{aligned} &\tilde{z}_2 = 2w^{1/2}, \quad \tilde{u} = C2^{-\alpha}, \quad \tilde{y}_2 z_2^{-1} \sim \beta, \quad \tilde{R} = D + I + CJ \\ &-\tilde{y}_1\tilde{z}_2 \sim 2(1-\alpha)rw^{(1-\alpha)/2}\tilde{R} \\ &\tilde{z}_2 z_2'/2 = -\tilde{y}_1\tilde{y}_2 \sim 2\beta(1-\alpha)rw^{(1-\alpha)/2}\tilde{R}, \quad u' \sim (\alpha-1)2^{-\alpha}\beta_4 \int w^{-\alpha}\tilde{R}dr \end{aligned} \tag{4.13}$$

where the tilde denotes that in the expressions for τ_i the terms $\tau_{ij} \in L$ with reduced order of the pole with respect to r and $r = \infty$ are omitted: these are such that $\nu_\infty(\tau_{ij}) - \nu_\infty(\tau_i) \geqslant 1 - \alpha$; for u' the corresponding expression is obtained taking into account Criterion 2; for $f(h, h_1, h_2, u, R, r, \varepsilon)$ here and below we will denote by f, f', f'',... the derivatives with respect to ε of the general solution

$$\frac{1}{i!}\frac{\partial^i f}{\partial \varepsilon^i}(h, h_1, h_2, u(h, h_1, h_2, C, D, r, \varepsilon),\ R(h, h_1, h_2, C, D, r, \varepsilon),\ r, \varepsilon)\Big|_{\varepsilon=0}$$

$$i = 0, 1, 2, \ldots$$

where the quantities z_2', z_1', y_1' are expressed from relations (4.3).

Differentiating (4.4) with respect to ε when $\varepsilon = 0$ and substituting expressions (4.13) into its right-hand side, we obtain

$$dR'/dr \sim (\alpha-1)\beta w^{-(\alpha+1)/2}\tilde{R}^2[\alpha - 2 - (1+\alpha)(1-\alpha)^2 r^2 w^{-1}] \tag{4.14}$$

Consequently, $\nu_\infty(dR'/dr) \geqslant 1 - \alpha$ and the terms neglected in (4.14) lie in $L_2 = \{f \in L \,|\, \nu_\infty f \geqslant 2\}$

8. *The case* $\alpha \in -1-2N_0$, $\beta_4 = 0$. It follows from Criteria 1 and 2 that $\tilde{R}, u' \in L$, and consequently, an additional algebraic ε^1-integral $u + \varepsilon F_1(u, R, r)$ exists.

Proposition 5. Under the conditions of this case R' belongs to L.

Proof. From formula (4.14), according to Criterion 2, since $-1 \notin [(\alpha-3)/2, -2-j]$ $(j = 0, 1)$, we obtain that terms in R' that are linear in D belong to L. It is obvious that the coefficients of D^2 in R' belong to L. For the coefficients of h_i^2, integrating by parts, we have

$$\begin{aligned} &\int w^{-j-(\alpha+1)/2}\left(\int w^{(\alpha-3)/2}dr\right)^2 dr = J_j \int w^{(\alpha-3)/2}dr - \int w^{(\alpha-3)/2} J_j dr \\ &J_j = \int w^{-j-(\alpha+1)/2}\left(\int w^{(\alpha-3)/2}dr\right)dr \end{aligned} \tag{4.15}$$

By Criterion 2 we have $J_j \in L$; $\nu_\infty(dJ/dr) = 2j + (\alpha+1)$, with the exception of $\{j = 1, \alpha = -1\}$, by property 3, $\nu_\infty J j = 2j = \alpha$ and, by Criterion 1, since $\nu_\infty(w^{(\alpha-3)/2}J) \geqslant 3$, the last integral in (4.15) belongs to L. When $\alpha = -1$, by Lemma 1, $\nu_\infty J_1 = 0$ and, in view of Criterion 1

$$\int w^{-2} J_1 dr \in L$$

For the coefficients of h_1h_2 and h_2^2 these estimates are satisfied even more.

The subcase $\alpha \neq -1$, $h_1h_2 \neq 0$. In view of Propositions 1 and 3, the coefficient of C^0D^0 in u'' is calculated by substituting $u = 0$ in the first equation of system (4.4). It is equal to

$$2^{-\alpha-2}(1-\alpha)h_2\int w^{-1-\alpha}Idr$$

By Criterion 2 this integral contains lnw if and only if $-1 \in [(\alpha - 3)/2, -1-(\alpha + 3)/2]$. Under the conditions of this subcase -1 belongs to this interval. We obtain a contradiction with the second Husson identity.

9. *The case* $\alpha \in -N_0$. Substituting into the right-hand side of (4.8) the terms of maximum power in D (the result of this substitution is denoted below by a tilde) in the quantities indicated below

$$\tilde{u}=2^{-\alpha}C,\quad \tilde{R}\sim D,\quad u'\sim(\alpha-1)2^{-\alpha}D\left(\beta_4 I_0+2\beta C I_1+\frac{1}{2}\alpha C^2 I_2\right)$$

$$R'\sim(\alpha-1)D^2\beta\{(\alpha-2)I_1-(\alpha+1)(\alpha-1)^2 I_3\}-\frac{1}{2}D^2C(\alpha-1)\{(1-\alpha)rw^{-1}+\alpha I_2\}$$

$$I_0=\int w^{-\alpha}dr,\quad I_1=\int w^{-(\alpha+1)/2}dr,\quad I_2=\int w^{-1}dr,\quad I_3=\int r^2w^{-(\alpha+5)/2}dr$$

we obtain, in view of Propositions 1 and 3, the coefficient of D^2 in du''/dr in the two subcases below, using analytical calculations in MAPLE V release 3.

The subcase $\alpha = -1, \beta_4 = 0, \beta \neq 0, (h_1, h_2) \neq 0$. The coefficient of D^2C^2 in du''/dr is equal to $-12\beta rw^{-1}$. This contradicts the algebraic relation (4.10) since $\tilde{u}, \tilde{R}, u', R' \in L$, see case 8.

The subcase $\alpha \in -N_0, \beta = 0, \beta_4 \neq 0, h_2 \neq 0$. The inclusion $u' \in L$ follows from (4.7), and the inclusion $R' \in L$ follows from (4.14). The coefficient of D^2C in du''/dr, in view of the fact that

$$\int w^{-\alpha}I_2dr=I_0I_2-\int w^{-1}I_0dr$$

is equal to

$$2^{-\alpha}\alpha(\alpha-1)^2\beta_4\left(\int rw^{-1-\alpha}dr+\frac{3}{2}\int w^{-1}I_0dr\right)$$

where the first integral is algebraic when $\alpha \neq 0$, while the second contains ln w when $\alpha \in -N_0$. When $\alpha \neq 0$ this contradicts the second Husson equality (4.10) (here we distinguish the third Clebsch case $\alpha = 0$).

10. *The case* $\beta = h_1 = 0, \alpha \in 1/2 - N_0$ In this case the non-uniqueness condition 3 in (4.6) is satisfied, and hence we have two functionally independent supplementary algebraic ε^0-integrals: $u, v:= R - 2^{\alpha}uJ$. In system (4.4) we will change to the variables u, v, r. It follows from the existence of a supplementary algebraic ε^1-integral $F^0(u,v) + \varepsilon F^1(u, v, r)$ that

$$\frac{\partial F^0}{\partial u}(2^{-\alpha}C,D)u'+\frac{\partial F^0}{\partial v}(2^{-\alpha}C,D)\omega'+F^1(2^{-\alpha}C,D,r)=\text{const} \tag{4.16}$$

$$\left(\frac{\partial F^0}{\partial u},\frac{\partial F^0}{\partial v}\right)\not\equiv 0$$

The subcase $\beta_3 \neq 0$. Expansion (4.7) takes the form

$$\frac{u'}{\alpha-1}=2^{-\alpha}\int\left(-\beta_3 w^{-\alpha}+\frac{\alpha}{2}C^2w^{-1}\right)(D+CJ)dr \tag{4.17}$$

where only the coefficient of D^1C^0 does not belong to L and contains ln σ, the coefficient of D^0 belongs to $K \subset L$. From (4.4) we have

$$\frac{1}{\alpha-1}\frac{d\omega'}{dr}=w^{-2}C\left(\frac{h_2}{4}(\alpha-3)Cw^{\alpha-1}+(\alpha-1)r\tilde{R}\right)\times$$
$$\times\left(\frac{h_2}{4}Cw^{\alpha-1}-(\alpha-1)r\tilde{R}\right)-\frac{C}{2}w^{-1}\tilde{R}^2-\frac{2^{\alpha}}{\alpha-1}J\frac{du'}{dr},\quad \tilde{R}=D+CJ \tag{4.18}$$

where u' is given by (4.17). All the terms on the right-hand side of (4.18), with the exception of the last one, and the coefficients of DC^2 and D^0C^3 in the last term belong to L_2. Hence

$$\frac{1}{\alpha-1}\frac{d\omega'}{dr}=\beta_3 w^{-\alpha}J(D+CJ) \mod L_2 \tag{4.19}$$

By Criterion 1 the primitive of the coefficient of D in this equation mod L is equal to

$$\beta_3 A_0 \int rw^{-1}dr \sim \ln w$$

and for C it is equal to

$$-\frac{1}{\alpha-1}\beta_3 A_0 A_n \int w^{-1/2}dr \sim \ln\sigma, \quad \alpha=\frac{1}{2}-n$$

where

$$\int w^{\alpha-2}dr = r\sum_{i=0}^{n} A_i w^{i-n-1/2}, \quad A_0=\frac{1}{2n+1}, \quad A_n=\frac{2^n n! h^{-n}}{(2n+1)!!}$$

The coefficients of $\int rw^{-1}dr$ and $\int w^{-1/2}dr$ on the left-hand side of identity (4.16) are equal to, respectively

$$\frac{(\alpha-1)^2}{2}\beta_3 A_0 h_2 D\frac{\partial F^0}{\partial v}(2^{-\alpha}C, D)$$
$$-(\alpha-1)\beta_3\left[\frac{1}{2^\alpha}\tilde{B}D\frac{\partial F^0}{\partial u}+\frac{\alpha-1}{4}A_0A_nh_2^2C\frac{\partial F^0}{\partial v}\right] \tag{4.20}$$

where $\tilde{B}$ is the coefficient of $\int w^{-1/2}dr$ in $\int w^{n-1/2}dr$, and the coefficients given by (4.20) should vanish.

We will use the fact that the constants C and D are arbitrary. When $hh_2 \neq 0$ we obtain $\beta_3 = 0$. When $h_2 = 0$ and $\beta_3 h \neq 0$ we have $\partial F^0/\partial u \equiv 0$ and Eq. (4.18) reduces to

$$\frac{\omega'}{\alpha-1}=-CD^2\int\left(\frac{(1-\alpha)^2r^2}{w^2}+\frac{w^{-1}}{2}\right)dr \equiv -\frac{\alpha}{2}CD^2\int\frac{dr}{w}\mod K \tag{4.21}$$

whence $\partial F^0/\partial u \equiv 0$, i.e. we obtain a contradiction with (4.16).

The subcase $\beta_3 = 0$. It follows from (4.17), (4.19) and (4.20) that $u', \omega' \in L$.

We will prove successively that the following functions belong to the ring L or the set $w^{1/4}L$ and we will estimate their v_∞: $v_\infty r = v_\infty w^{1/2} = v_\infty \tilde{z}_2 = -1$; $v_\infty \tilde{u} = v_\infty J = v_\infty \tilde{v} = v_\infty \tilde{R} = 0$ [24], $v_\infty \tilde{z}_1 = 1$; using Properties 1 and 2 we have $v_\infty \tilde{p}_3 = v_\infty \tilde{y}_1 = \alpha - 1$; from (4.3) $v_\infty z_2' = v_\infty u' = 0$, $v_\infty y_2' = \alpha$; in view of (4.19) and Criterion 1 we have $v_\infty \upsilon' \geqslant 0$; since the coefficient of CD^2 in υ' is equal to (4.21), while in ν' it is zero, we have

$$v_\infty\omega' = v_\infty R' = 0, \quad v_\infty p_3' = v_\infty y_1' = \alpha-1, \quad v_\infty z_1' = 2\alpha-1, \quad v_\infty z_2'' \geqslant 0$$

From the second equation of (4.4) we obtain

$$\frac{1}{4(\alpha-1)}\frac{dR''}{dr} = h_2 w^{(\alpha-1)/2}(uz_2^{\alpha-3})'' + (1-\alpha)r[R(z_2^{-2}-\frac{1}{4}w^{-1})]'' -$$
$$-w^{(1-\alpha)/2}(R^2uz_2^{\alpha-3})' + [h_2 w^{(\alpha-1)/2}\tilde{u}\tilde{z}_2^{\alpha-1} + (1-\alpha)r\tilde{R}]h_2^2\tilde{z}_2^{-6} \tag{4.22}$$

Since all terms on the right-hand side of (4.22), apart from $h_2 w^{(\alpha-1)/2}u''\tilde{z}_2^{\alpha-3}$, belong to L_2, the quantity

$$\frac{d\omega''}{dr}+2^\alpha J\frac{du''}{dr}=\frac{dR''}{dr}-2^\alpha\frac{dJ}{dr}u''=\frac{dR''}{dr}-4(\alpha-1)h_2w^{(\alpha-1)/2}\tilde{z}_2^{\alpha-3}u''$$

belongs to L_2.

The subcase $\beta_3 = 0$, $h_2 \neq 0$. It following from system (4.4) that

$$\frac{1}{\alpha-1}\frac{du''}{dr}-2h_2w^{(1-\alpha)/2}\tilde{R}\tilde{z}_2^{-\alpha-3}=4\alpha w^{(1-\alpha)/2}(u^2Rz_2^{\alpha-3})' \in L_2$$

whence, in view of Criterion 1

$$u'' = -2^{-2-\alpha}(\alpha-1)h_2 \int w^{-1-\alpha}(D+CJ)dr \bmod L$$
$$\omega'' = \frac{(\alpha-1)h_2}{4}\int w^{-1-\alpha}J(D+CJ)dr \bmod L$$

From the existence of the supplementary algebraic ε^2-integral $F^0(u, v) + \varepsilon F^1(u, v, r) + \varepsilon^2 F^2(u, v, r)$ it would follow that

$$\frac{\partial F^0}{\partial u}(2^{-\alpha}C, D)u'' + \frac{\partial F^0}{\partial v}(2^{-\alpha}C, D)\omega'' \in \overline{C(r)}[\ln v], \left(\frac{\partial F^0}{\partial u}, \frac{\partial F^0}{\partial v}\right) \neq 0 \tag{4.23}$$

where $\overline{C(r)}$ [ln v] is a ring above the field $\overline{C(r)}$, generated by the element ln v.

But integration by parts and the use of well-known formulae [24] show that

$$\frac{u''}{h_2} = -2^{-\alpha-2}\left\{(\alpha-1)DB\ln\sigma + C\frac{A_1\ln w}{2}\right\} \bmod L$$
$$\frac{\omega''}{h_2} = \frac{1}{4}\left\{D\frac{A_1\ln w}{2} + C\frac{A_1A_n\ln\sigma}{(\alpha-1)^{1/2}}\right\} \bmod L$$
$$B = \begin{cases} 0 & , n=0 \\ (\alpha-1)^{-1/2}2^{1-n}(2n-3)!!h^{n-1}/(n-1)! & , n\in N \end{cases}, \quad A_1 = \frac{2}{h(4n^2-1)}$$

It follows from (4.23) that

$$\begin{vmatrix} CA_1/(\alpha-1) & DB \\ DA_1/(2(\alpha-1)) & CA_1A_n/(\alpha-1)^{3/2} \end{vmatrix} = 0$$

whence, in view of the fact that C and D are arbitrary, it follows that $A_1A_n = A_1B = 0$, i.e. $A_1 = 0$ (here we distinguish the Chaplygin case $\alpha = 1/2$, $\beta_1 = \beta = \beta_3 = h_1 = 0$ or $A_n = B = 0$, which corresponds to an empty set of the parameters of the problem.

5. THE REPLACEMENT $(y_1, y_2, p_3) \to \varepsilon(y_1, y_2, p_3)$

In the remaining case $\beta_1 = \beta = \beta_3 = H_1 = H_2 = 0$, $\alpha \in -N_0$ we introduce a small parameter ε into system (3.1), (3.2) by making this replacement. We have

$$\frac{dy_1}{dz_1} = \frac{\alpha r y_1 - p_3z_2/2}{rz_1 - \varepsilon y_1p_3}, \quad \frac{dy_2}{dz_1} = \frac{-\alpha r y_2 + p_3z_1/2}{rz_1 - \varepsilon y_1p_3} \tag{5.1}$$
$$p_3 = \frac{y_1z_2 + y_2z_1}{2(\alpha-1)r}, \ (\alpha-1)r^2 = h + \frac{1}{4}(z_1^2+z_2^2) - \varepsilon y_1y_2, \quad z_2 = -\varepsilon p_3^2 z_1^{-1}$$

where h is the constant of the integral H.

Since system (5.1) has two algebraic ε^0-integrals

$$u = y_1z_1^{-\alpha}, \quad v = y_2z_1^{\alpha}w^{-1/2}, \quad w = 4h + z_1^2 \tag{5.2}$$

we can change to osculating algebraic coordinates (h, u, v, z_1)

$$\dot{u} = \varepsilon f_1(h,u,v,z_1) + \varepsilon^2 f_2(h,u,v,z_1) + \dots \tag{5.3}$$
$$\dot{v} = \varepsilon g_1(h,u,v,z_1) + \varepsilon^2 g_2(h,u,v,z_1) + \dots$$
$$f_1 = \frac{v}{\alpha-1}[v^2z_1^{1-4\alpha} + 2\alpha(\alpha-1)u^2z_1^{-1}]w^{-1/2}$$

$$g_1 = \frac{uv^2}{1-\alpha}[2\alpha^2 w - 8\alpha w + 24\alpha h + 7(w-4h)]z_1^{-1}w^{-\frac{1}{2}},\dots$$

Expressions for f_1, g_1, f_2 and g_2 were calculated in MAPLE V.

Suppose $\mathcal{F}(y_1, y_2, r, z_1, z_2, p_3)$ is the supplementary algebraic interval of system (3.1) when $H_1 = H_2 = 0$.

Then

$$\mathcal{F}(\sqrt{\varepsilon}y_1, \sqrt{\varepsilon}y_2, r, z_1, z_2, \sqrt{\varepsilon}p_3) = \sum_{l=l_0}^{\infty} \varepsilon^{l/s}F^{l/s}(h, u, v, z_1), \quad l_0 \in \mathbf{Z}, \quad s \in \mathbf{N}$$

is the algebraic integral (5.1) with arbitrary ε. Suppose m is a minimum number such that the quantity $F^{m/s}$ is functionally independent of h. Then the integral $\sum_{l=l_0}^{\infty} \varepsilon^{l/s}F^{l/s}$ can be replaced by $\sum_{l=m}^{\infty} \varepsilon^{(l-m)/s}F^{l/s}$. In view of the fact that system (5.1) is rational with respect to ε its component for integer powers of ε will also be the integral (5.1).

It follows from the existence of the supplementary algebraic ε^2-integral $F^0(u,v) + \varepsilon F^1(u,v,z_1) + \varepsilon^2F^2(u, v, z_1)$ of system (5.1) that

$$-\frac{\partial F^1}{\partial z_1} = \frac{\partial F^0}{\partial u}(u,v)f_1(u,v,z_1) + \frac{\partial F^0}{\partial v}(u,v)g_1(u,v,z_1), \quad \left(\frac{\partial F^0}{\partial u}, \frac{\partial F^0}{\partial v}\right) \not\equiv 0 \tag{5.4}$$

$$\begin{aligned}-\frac{\partial F^2}{\partial z_1} &= \frac{\partial F^0}{\partial u}(u,v)f_2(u,v,z_1) + \frac{\partial F^0}{\partial v}(u,v)g_2(u,v,z_1) + \\ &+ \frac{\partial F^1}{\partial u}(u,v,z_1)f_1(u,v,z_1) + \frac{\partial F^1}{\partial v}(u,v,z_1)g_1(u,v,z_1)\end{aligned} \tag{5.5}$$

Here and below, for brevity, the dependence on h is not indicated.

It follows from relations (5.3) and (5.4) that

$$\mathrm{res}_{z_1=0}\left(-\frac{\partial F^1}{\partial z_1}\right) = -\alpha h^{-\frac{1}{2}}uv, \left(v\frac{\partial F^0}{\partial v} - u\frac{\partial F^0}{\partial u}\right) \equiv 0$$

whence, when $\alpha \neq 0$ (here we distinguish the third Clebsch case $\alpha = 0$), it follows that $F^0 = F^0(uv)$, and, without loss of generality, we can put $F^0 = uv$. Then, in view of (5.4)

$$F^1 = \frac{v^2}{1-\alpha}\left[v^2 w^{\frac{1}{2}}\left(\sum_{k=0}^{-2\alpha} \frac{C_{-2\alpha}^k}{2k+1}(-4h)^{-2\alpha-k}w^k\right) - u^2(6\alpha-7)w^{-\frac{1}{2}}\right] + \varphi^1(u,v)$$

Suppose $C(z_1, w^{1/2})$ is the space of rational functions of $z_1, w^{1/2}, \delta = 2h^{1/2} + w^{1/2}$. Suppose N is a linear space above C, generated by $C(z_1, w^{1/2})$, $\ln z_1$, $\ln\delta$.

In view of the fact that the primitives $\int f_1 dz_1$ and $g_1 dz_1$ belong to N, the term φ^1 makes no contribution to $-F^2 \bmod N$. To calculate $-F^2 \bmod N$ it is sufficient [24] to substitute into relations (5.5) terms of small powers of w

$$\frac{\partial(F^1 - \varphi^1)}{\partial u} = \frac{2(6\alpha-7)}{\alpha-1}uv^2w^{-\frac{1}{2}}(1 + O(w))$$

$$\frac{\partial(F^1 - \varphi^1)}{\partial v} = \frac{2v\,w^{-\frac{1}{2}}}{\alpha-1}[(6\alpha-7)u^2 - 2v^2(4h)^{-2\alpha}w + O(w^2)]$$

(the remaining terms make no contribution to the residues at the point $w = 0$). Taking into account the fact that

$$vf_2 + ug_2 = \frac{2}{(\alpha-1)^2}uv^3z_1w^{-2}[6(3\alpha^2 - 7\alpha + 4)u^2 + ((5\alpha-6)w + 2h)v^2]$$

we obtain from relations (5.5) that $-F^2$ mod$N$ is a polynomial in u and v and its coefficient of uv^5 is equal to $4h^{-2d}(d-1)^{-2}\ln w$, which contradicts the fact that F^2 is algebraic.

Remark. When there is a complete set of algebraic ε^0- integrals, instead of obstacles to integrability of the Husson form (4.16); (4.10), (4.23); ... in this problem we use the equivalent obstacles—the residues and periods of the right-hand sides of (5.4), (5.5)..., multiplied by dz_1, on the Riemann surface (z_1, $\tilde{y}_1(z_1), \tilde{y}_2(z_1)$), defined by relations (5.2) (on which they are rational). In particular, a smaller number of quadratures is required to calculate these obstacles.

6. PROOF OF THE ASSERTIONS OF SECTION 2

Proof of Property 1. Suppose

$$g=\sum_{l=0}^{n_1} g_l \ln^l \vartheta$$

Then

$$\nu_\infty(f+g)=\min_l \operatorname{ord}_\infty(f_l+g_l)\geqslant \min_l\{\operatorname{ord}_\infty f_l, \operatorname{ord}_\infty g_l)\}=\min(\nu_\infty f, \nu_\infty g\}$$

Proof of Property 2. Obviously

$$\operatorname{ord}_\infty(f_l g_m)=\operatorname{ord}_\infty f_l+\operatorname{ord}_\infty g_m\geqslant \nu_\infty f+\nu_\infty g,\quad 0\leqslant l\leqslant n,\quad 0\leqslant m\leqslant n_1$$

Suppose l_0 and m_0 are minimum numbers such that

$$\operatorname{ord}_\infty f_{l_0}=\nu_\infty f,\quad \operatorname{ord}_\infty g_{m_0}=\nu_\infty g$$

Then

$$\operatorname{ord}_\infty\left(\left(\sum_{l=0}^{n} f_l \ln^l\vartheta\right)\left(\sum_{l=0}^{n_1} g_m \ln^m\vartheta\right)\right)_{l_0+m_0}=\operatorname{ord}_\infty \sum_{l=-l_0}^{m_0} f_{l_0+l}g_{m_0-l}=\operatorname{ord}_\infty(f_{l_0}g_{m_0})=\nu_\infty f+\nu_\infty g$$

Proof of Lemma 1. Using a Puiseux expansion

$$g'(s)=s^{-k}(a_0+a_1 s^{-1/t}+\ldots),\quad a_0\neq 0,\quad t\in N,\quad k\in \mathbf{Q}/1$$

we have

$$g(s)=C+s^{1-k}[a_0/(1-k)-a_1 s^{-1/t}/(1-k-1/t)+\ldots],\quad C=\text{const}$$

whence

$$\operatorname{ord}_\infty g=\min\{\operatorname{ord}_\infty C, \operatorname{ord}_\infty g'-1\},\quad \operatorname{ord}_\infty 0=+\infty$$

Consequently, when $\operatorname{ord}_\infty g<0$ we obtain assertion a, when $\operatorname{ord}_\infty g<0$ we have $\operatorname{ord}_\infty g'>1$ and $C\neq 0$ b, and when $\operatorname{ord}_\infty g'>0$ we have $C\neq 0$ a.

Proof of Property 3a. Actually

$$\begin{aligned}\nu_\infty f'&=\min_l \operatorname{ord}_\infty(f_l'+2(l+1)W^{-1}f_{l+1})\geqslant \min_l\{\operatorname{ord}_\infty f_l', \operatorname{ord}_\infty(W^{-1}f_{l+1})\}\geqslant 1+\\&+\min_l \operatorname{ord}_\infty f_l=1+\nu_\infty f\end{aligned}$$

where the first equality follows from the definition of ν_∞, the first inequality follows from Property 1, and the second follows from Lemma 1.

Proof of Property 3b. Suppose the number l is such that $\nu_\infty f=\operatorname{ord}_\infty f_l$ Then

$$\nu_\infty f'\leqslant \operatorname{ord}_\infty(f_l'+2(l+1)W^{-1}f_{l+1})=\operatorname{ord}_\infty f_l'=1+\operatorname{ord}_\infty f_l=1+\nu_\infty f$$

where the inequality follows from the definition of ν_∞, the second equality follows from Lemma 1, since possibility b in it is eliminated, and the first equality follows from the second and from the relation

$$\mathrm{ord}_\infty(f_{l+1}W^{-1}) = 2 + \mathrm{ord}_\infty f_{l+1} \geqslant 1 + \mathrm{ord}_\infty f_l$$

Proof of Criterion 1. Suppose that either or $j = 0, k = -1, -3/2, -2, \ldots$, or $j = 1, k = -3/2, -2, -3/2, \ldots$. We put

$$I_{jkl} = \int s^j W^k \ln^l \vartheta ds, \quad l = 0, 1, 2, \ldots$$

Then $I_{jk0} - \delta_{0j} B_k \ln \vartheta \in K$ for a certain constant B_k and hence $I_{jk0} \in L$. By making the replacement $s = 1/z$ we can establish that I_{jkl} is an analytic function at the point $s = \infty$. Consequently, $\nu_\infty I_{jk0} \geqslant 0$.

Integrating by parts

$$I_{jkl} = I_{jk0} \ln^l \vartheta - 2l \int W^{-1} I_{jk0} \ln^{l-1} \vartheta ds \tag{6.1}$$

we use induction with respect to l, and inside it induction with respect to k. We consider four cases of the evenness of the quantities j and $2k$. For example, if $j = 0$ and k is an integer, we have

$$I_{0k0} = B_k \ln \vartheta + s \sum_{m=k+1}^{-1} A_m W^m, \quad A_m = \mathrm{const}$$

Substituting this expression into (6.1) we obtain

$$(1 + 2lB_{-1}) I_{0-1l} = I_{0-10} \ln^l \vartheta - 2l \sum_{m=k+1}^{-1} A_m I_{1,m-1,l-1}$$

where the second term belongs to $L_0 = \{f \in L \,|\, \nu_\infty f \geqslant 0\}$ by the assumption of the induction.

Proof of Criterion 2. If $l, m < 0$, we have $J_{lm} \in L$ in view Criterion 1. If $l < 0 \leqslant m$, we have

$$J_{lm} = -\sum_{i=l}^{m+l} A_i \int W^i dW \notin L \Leftrightarrow l \leqslant -1 \leqslant m + l$$

The case $m < 0 \leqslant l$ reduces to the previous integration by parts.

I wish to thank V. V. Kozlov for suggesting the problem and for useful advice, and Yu. A. Arkhangel'skii and D. V. Treshchev for their comments.

REFERENCES

1. STEKLOV, V. A., *The Motion of a Rigid Body in a Fluid*. Tip. Darre, Khar'kov, 1893.
2. KOZLOV, V. V. and ONISHCHENKO, D. A., The non-integrability of Kirchhoff's equations, *Dokl. Akad. Nauk SSSR*. 1982, **266**, 6, 1298–1300.
3. NOVIKOV, S. P. and SHMEL'TSER, I., Periodic solutions of Krichhoff's equations for the free motion of a rigid body in a fluid and extension of the Lyusternik–Shnirel'man–Morse (LSM) theory. I. *Funkst. Analiz i Yego Prilozheniya*, 1981, **15**, 3, 54–66.
4. KOZLOV, V. V., Two integrable problems of classical dynamics. *Vestnik MGU. Ser. 1, Mat. Mekh.*, 1981, **4**, 80–83.
5. KOZLOV, V. V., The non-integrability of Kirchhoff's equations. *Short Communications ICM*. Section 13, Warsaw, 1982, 41.
6. KOZLOV, V. V. and Treshchev, D. V., The non-integrability of the general problem of rotation of the dynamically symmetrical heavy rigid body with a fixed point. II. *Vestnik MGU. Ser. 1, Mat. Mekh.*, 1986, **1**, 39–44.
7. HUSSON E., Recherche des integrales algebraique dans les mouvement d'un solide pesant autor d'un point fixe. Ann. Fac. Sci. Univ. Toulouse. *Sci. Math. et Sci. Phys.* 1906, **8**, 73–152.
8. KIRCHHOFF, G. R., Über die Bewegung eines Rotationskörpers in einer Flussigkeit. *J. Reine und angew. Math.*, 1870, **71**, 237–262.
9. SEDOV, L. I., *Continuum Mechanics*. Vol. 2. Nauka, Moscow, 1973.
10. *Bogoyavlenskii*, O. I., Euler's equations in finite dimensional Lie coalgebras which arise in problems of mathematical physics. *Uspekhi Mat. Nauk*, 1992, **47**, 1, 107–146.
11. YAKH'YA Kh. M., New solutions of the problem of the motion of a gyrostat in potential and magnetic fields. *Vestnik MGU. Ser. 1. Mat. Mekh.*, 1985, **5**, 60–63.
12. SHABAT, B. V., *Introduction to Complex Analysis*. Nauka, Moscow, 1985.
13. CHAPLYGIN, S. A., *Selected Papers. Fluid Mechanics. Mathematics*. Nauka, Moscow, 1976.
14. ZIGLIN, S. L., Branching of the solutions and the non-existence of first integrals in Hamiltonian mechanics, I, II. *Funks. Analiz i Yego Prilozheniya*, 1982, **16**, 3, 30–41; 1983, **17**, 1, 8–23.
15. LIOUVILLE R., Aur le mouvement d'un corps solide pesant suspendu par l'un de ses points. *Acta math*. 1897, **20**, 239–284.

16. LYAPUNOV, A. M., A new case of the integrability of the differential equations of the motion of a rigid body in a fluid. In *Selected Papers*. Izd. Akad. Nauk SSSR, Moscow, 1954.
17. RUBANOVSKII, V. N., The quadratic integrals of the equations of motion of a rigid body in a fluid. *Prikl. Mat. Mekh.*, 1988, **52**, 3, 402–414.
18. CLEBSCH A., Über die Bewegung eines Körpers in eines Flussigkeit. *Mat. Ann.* 1870, **3**, 238–262.
19. GOLUBEV, V. V., P. L. Chebyshev's papers on the integration of algebraic functions. In *The Scientific Heritage of P. L. Chebyshev*. Izd. Akad. Nauk SSSR, Moscow, Leningrad, 1945.
20. KOZLOV, V. V., The non-existence of single-valued integrals and the branching of solutions in rigid-body dynamics. *Prikl. Mat. Mekh.*, 1978, **42**, 3, 400–406.
21. CHEBYSHEV, P. L., The integration of irrational differentials. In *The Complete Collected Papers of P. L. Chebyshev*, Vol. 2, 1947, 52–70.
22. ABEL N., *Oeuvres completes*, Vol. 2. Cristiania, Gröndahe, 1881.
23. ARKHANGEL'SKII, Yu. A., *Analytical Dynamics of a Rigid Body*. Nauka, Moscow, 1977.
24. BRYCHKOV, Yu. A., MARICHEV, O. I. and PRUDNIKOV, A. P., *Tables of Indefinite Integrals*. Nauka, Moscow, 1986.

Translated by R.C.G.